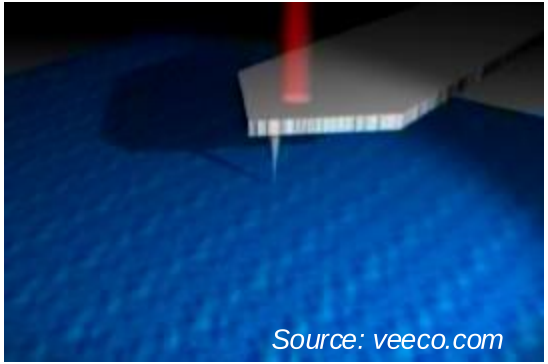
Lecture 13:
Beam deflection

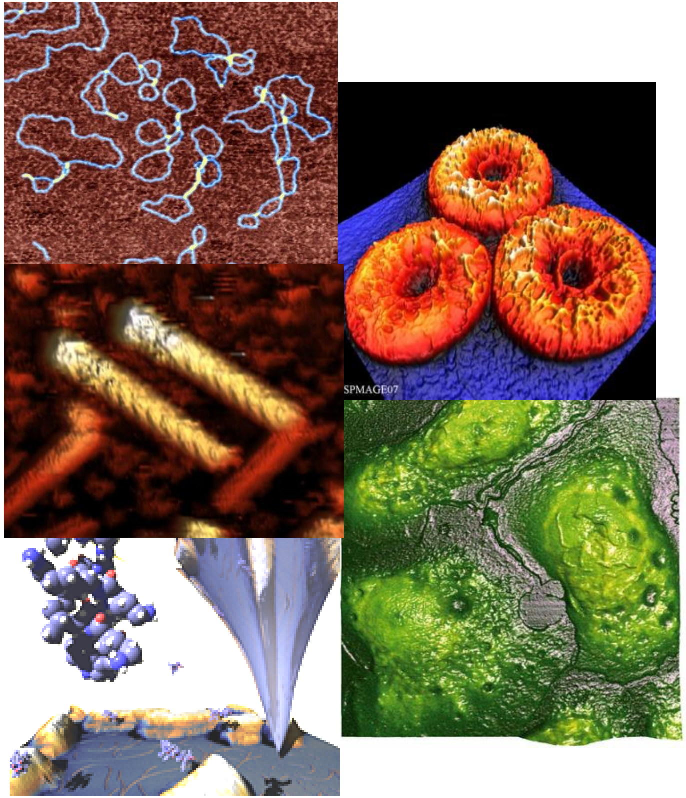
- Applications in nanotechnology
- Governing differential equations
- Solving beam deflection through integration
- Solving beam deflection through superposition
- Statically indeterminate beam deflection

Atomic Force Microscopy

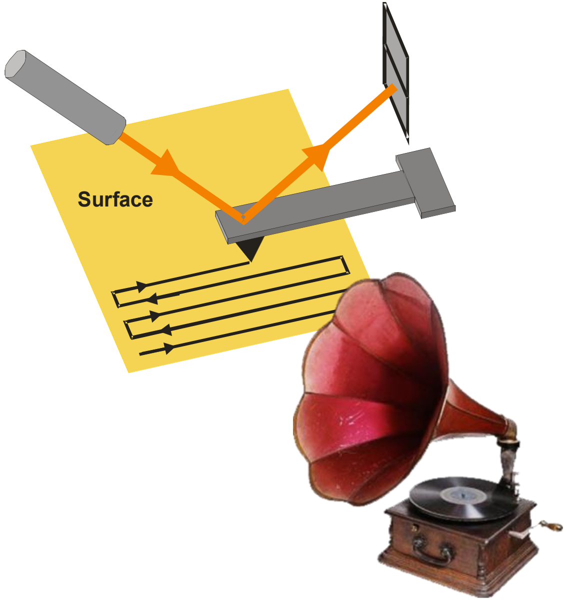
A versatile tool for nanoscale biology

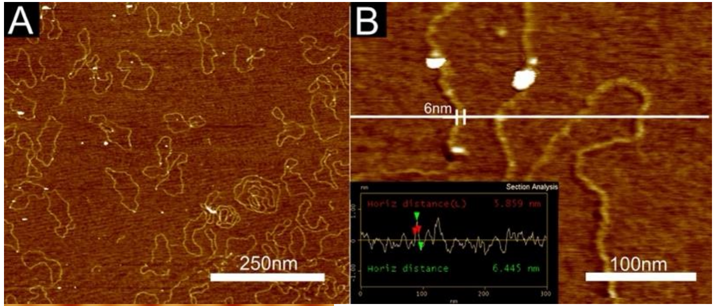


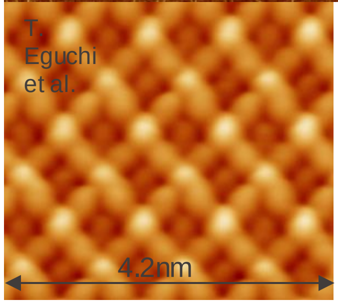
- Single molecule resolution
- High resolution imaging in aqueous solution
- Nanomanipulation
- Single molecule mechanics
- Imaging of living cells



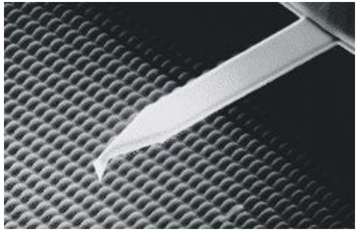
AFM: a Versatile Tool for Nanoscale Measurements







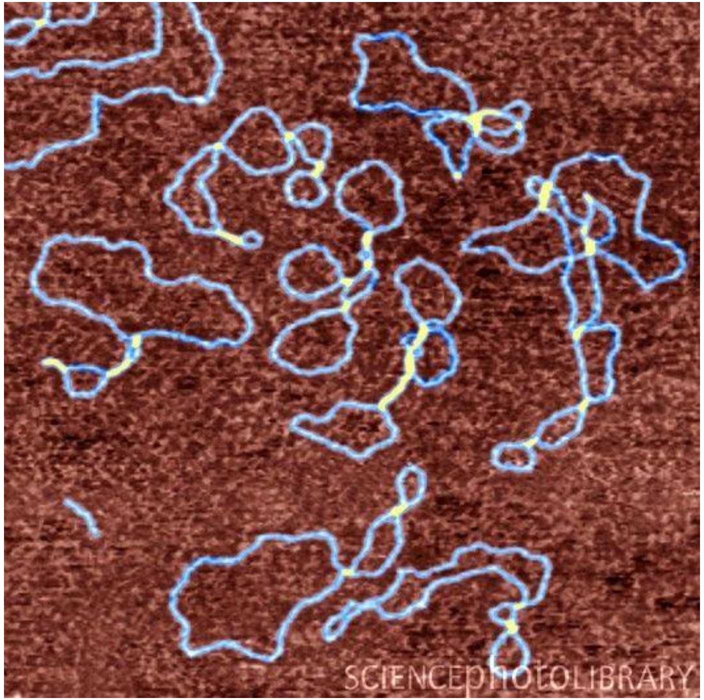
Cantilever



conductivity, surface potential, electrochemical potential, ion currents, magnetism, NMR….and many more

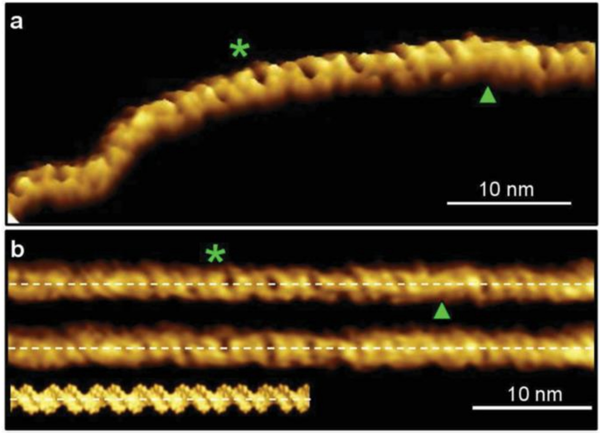
Single molecule resolution

Plasmid DNA on mica



Source: SciencePhotoLibrary

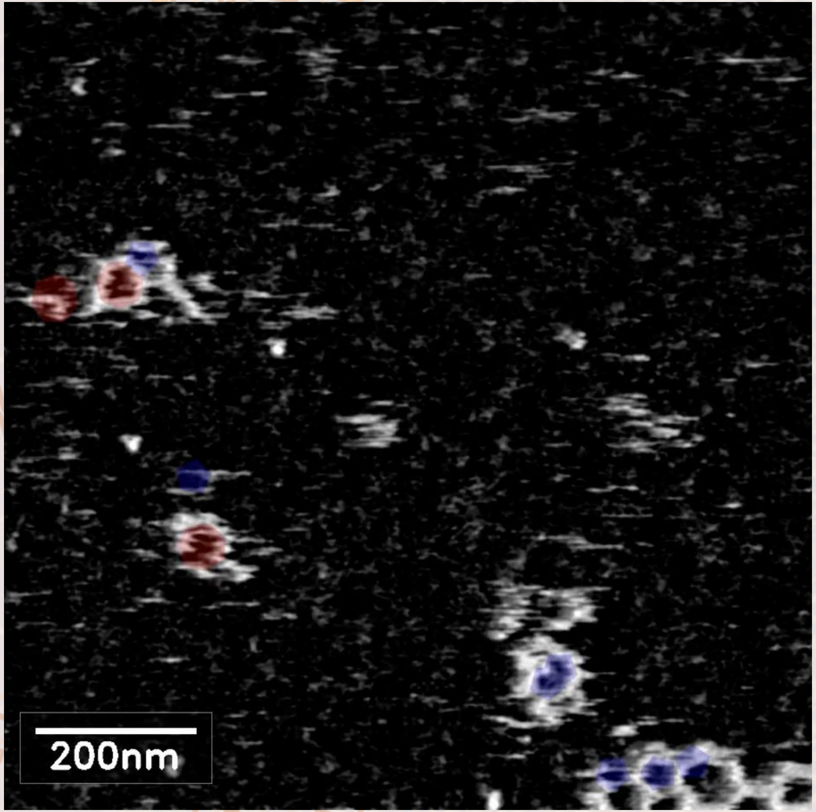
- Single molecules can be easily resolved
- Even the double helix!



Pyne et al. *Small, 10, Nr16, 2014*

Self-assembly and defect healing in DNA lattices

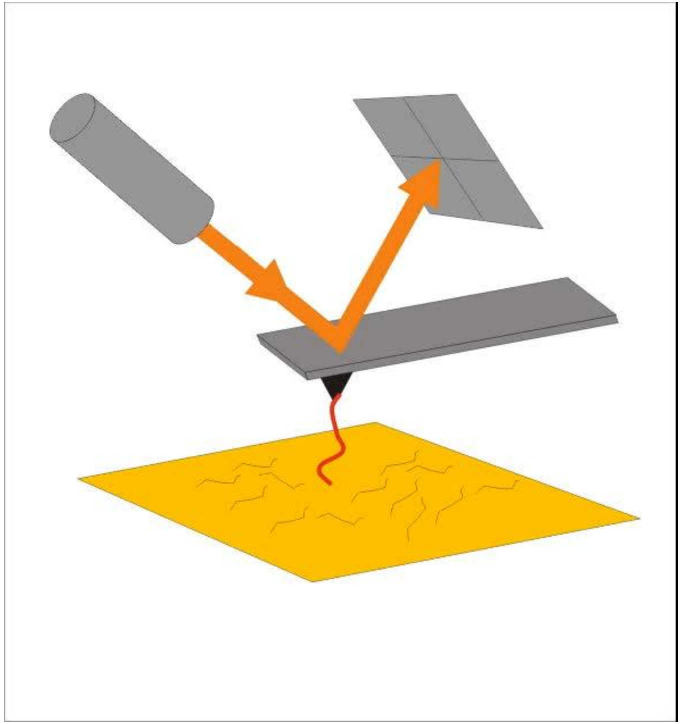
Defect healing in DNA-lattices

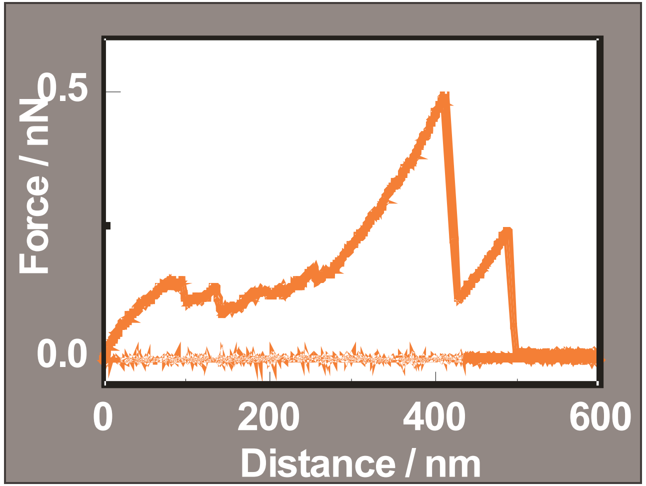


100kHz ORT rate, 80Hz line rate (512x512 pixel) (tube, XY=14um)

Nievergelt et al, *Small Methods* 2019

Single Molecule Force Spectroscopy





Force resolution: 10s of pN; limited by thermal motion of the cantilever

Single molecule force spectroscopy

Force curves as a tool for single molecule mechanics

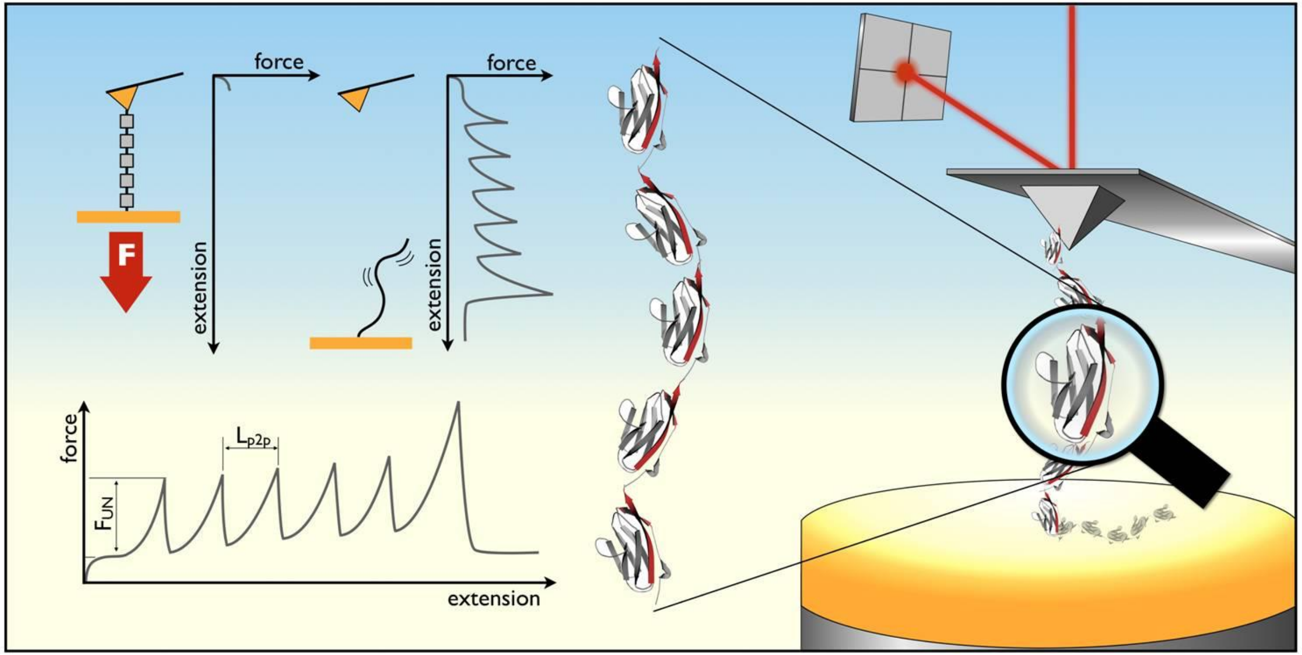
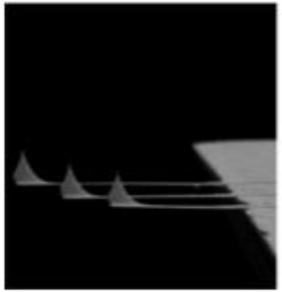


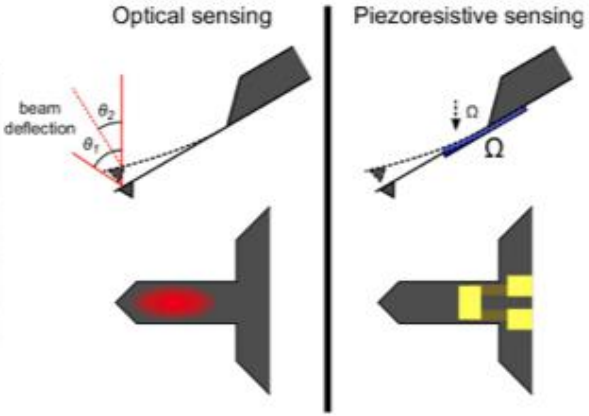
Image source: Dr Lorna Dougan - http://www1.mnp.leeds.ac.uk/ldougan/

AFM cantilever beam

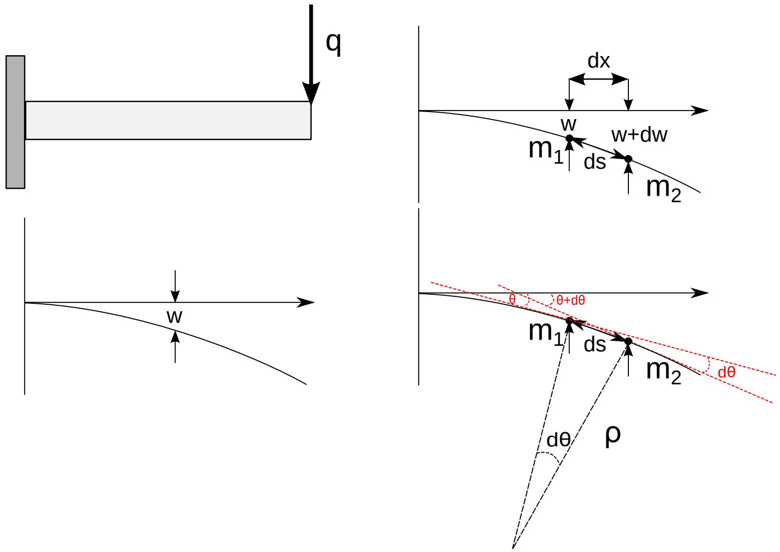
The gate to the nanoworld

- In order to measure very fine features, the cantilever probe needs to be very sharp and sensitive
- The deflection of the cantilever has to be measured very precisely
- Two methods are often used:
  - Optical beam deflection
  - Piezoelectric strain sensing





Beam bending



- We bend the cantilever beam by applying a load at the end
- w(x) describes the amount of deflection of the point on the cantilever from the zero axis
- Two points are a distance ds apart from each-other on the bent beam
- From this we can get a relationship that describes the curvature of the beam

Beam bending - Governing equation

$$\frac{dw}{dx} = -\theta \qquad (1)$$

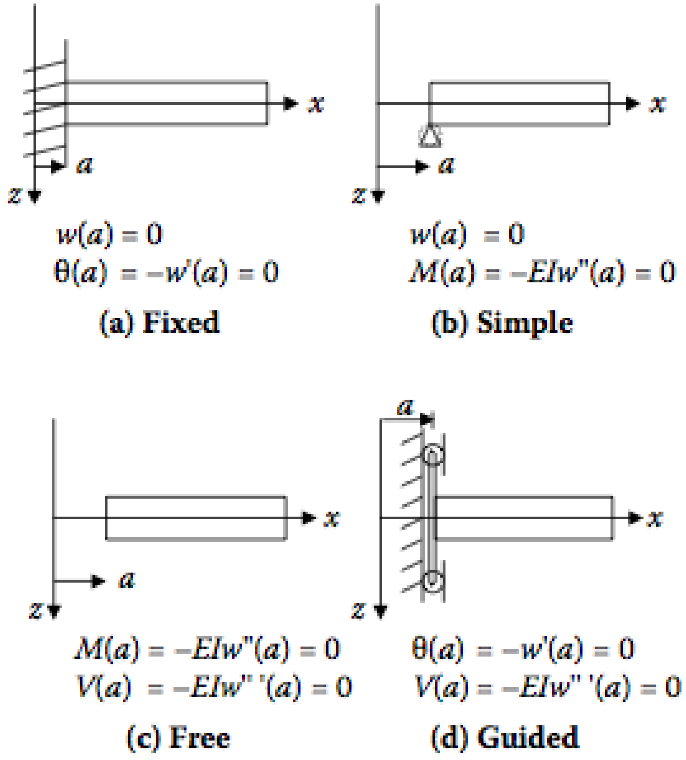
$$\frac{d^2w}{dx^2} = -\frac{M(x)}{EI} \qquad (2)$$

$$\frac{d^3w}{dx^3} = -\frac{V(x)}{EI} \qquad (3)$$

$$\frac{d^4w}{dx^4} = \frac{q(x)}{EI} \qquad (4)$$

We want to find a relationship between the beam deflection at a point x on the beam as a function of the load

We find 4 differential equations that relate loads to the deflection and the angle



Beam bending-Boundary condition

To solve for the beam bending equation through integration, we need boundary conditions

The type of support of the beam at its end determines the internal forces and moments at the ends, as well as its geometry

We have therefore two types of boundary conditions:

- Static boundary conditions: These come from static equilibrium and pertain to force related quantities (V,M)
- Kinematic boundary conditions: these define the deformational and geometric constraints for the angle and the bending

Beam bending - Abrupt changes

- When we have mathematical discontinuities due to an abrupt change in load or stiffness, we supplement our boundary conditions with the physical requirement that the neutral axis must be continuous!
- Deflection and tangent needs to be the same coming from both sides of the point of discontinuity:

$$\lim_{x \uparrow a} w(a) = \lim_{x \downarrow a} w(a)$$

$$\lim_{x \uparrow a} w'(a) = \lim_{x \downarrow a} w'(a)$$

Beam deflection

- Solving through integration

  - If we want to solve beam equation through integration, we need to integrate 4 times:

$$\int EIw''''(x)dx = \int q(x)dx$$

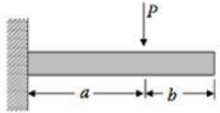
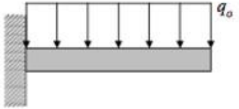
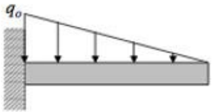
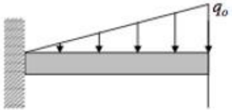
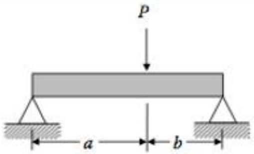
$$EIw(x) = \int\int\int\int q(x)dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

  - We know already that:

$$V = -\int q(x)dx + C_1 \qquad M = -\int q(x)dx + C_1x + C_2$$

  - Therefore:
    - We get C1 and C2 from the boundary conditions of M(x) and V(x)
    - We get C3 from the boundary condition of the angle of deflection and C4 from the boundary condition of w

Beam deflection – Solving through superposition

	$w(x)=\frac{PL^3}{6EI}\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]$
	$w(x)=\frac{Pa^3}{6EI}\left[3\left(\frac{x}{a}\right)^2-\left(\frac{x}{a}\right)^3\right] \quad 0\le x\le a$ $w(x)=\frac{Pa^3}{6EI}\left[3\left(\frac{x}{a}\right)-1\right] \quad a\le x\le L$
	$w(x)=\frac{q_oL^4}{24EI}\left[6\left(\frac{x}{L}\right)^2-4\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^4\right]$
	$w(x)=\frac{q_oL^4}{120EI}\left[10\left(\frac{x}{L}\right)^2-10\left(\frac{x}{L}\right)^3+5\left(\frac{x}{L}\right)^4-\left(\frac{x}{L}\right)^5\right]$
	$w(x)=\frac{q_oL^4}{120EI}\left[20\left(\frac{x}{L}\right)^2-10\left(\frac{x}{L}\right)^3+\left(\frac{x}{L}\right)^5\right]$
	$w(x)=\frac{PbL^2}{6EI}\left[\left(1-\left(\frac{b}{L}\right)^2\right)\frac{x}{L}-\left(\frac{x}{L}\right)^3\right] \quad 0\le x\le a$

- As long as the beams behave linearly elastic, we are dealing with linear differential equations.
- For such a situation, we can separate a difficult load profile into simpler sub parts:

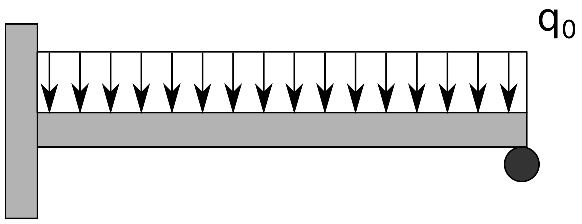
$$q(x) = q_1(x) + q_2(x) + \ldots$$

- We can then do the integrations over the individual q_i separately.
- To find the solution for the deflection due to the complex load profile, we can just sum up the deflections caused by the sub-loads q_i.

$$w(x) = \sum_i w_i(x)$$

- We can tabulate the deflection formulas due to standard loads.

Statically indeterminate beams – Solving through integration

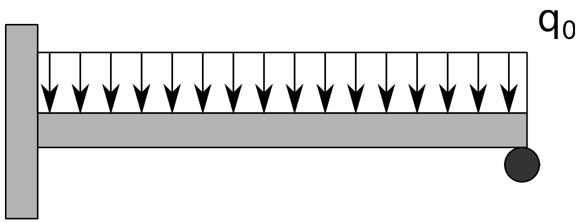


Often beams are supported more that absolutely required for static equilibrium.

A cantilever that is supported also on it's unmounted end is considered a "proper cantilever"

We treat over constrained beams in bending just like normal beams. The static indeterminacy is solved automatically through the use of the boundary conditions to calculate the integration constants.

Example: Statically indeterminate beams



- Solve the following statically indeterminate system through integration of the beam deflection differential equations. Calculate:
  - deflection
  - shear forces
  - bending moments
  - slopes

- Approach:
  - Set up load equation q(x)
  - Integrate the differential equations
  - Solve for the reaction forces using the boundary conditions

Beam Buckling

- Euler Buckling
- Effective length for buckling
- Effect of eccentricity

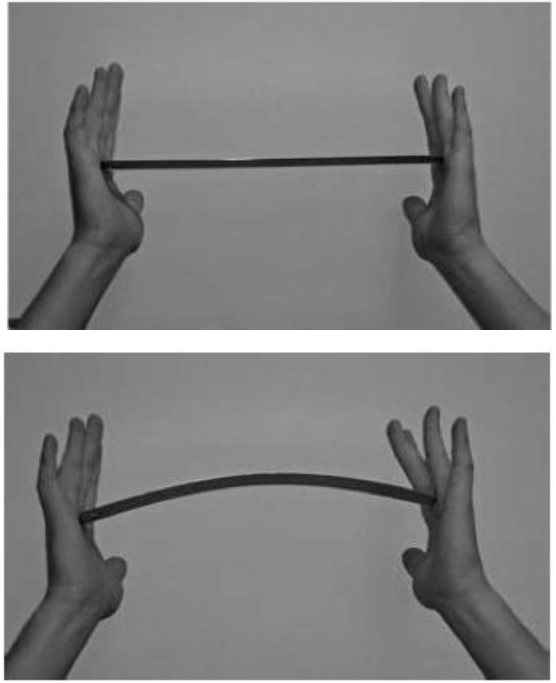
Stability- different criteria for resisting loads

Strength: the ability of a structure to withstand a load without the development of excessive stress

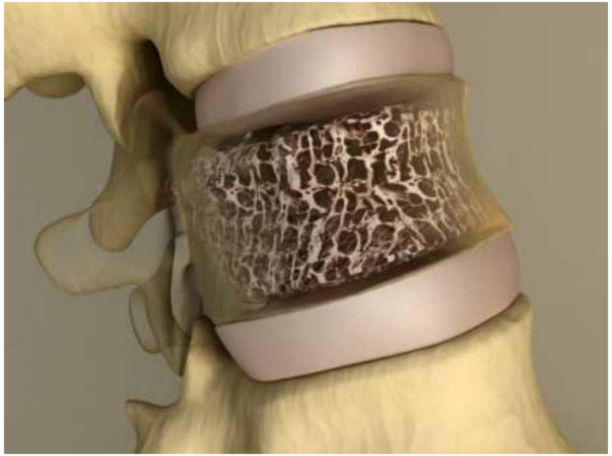
Stiffness: the ability of a structure to withstand a load without developing excessive deformation.

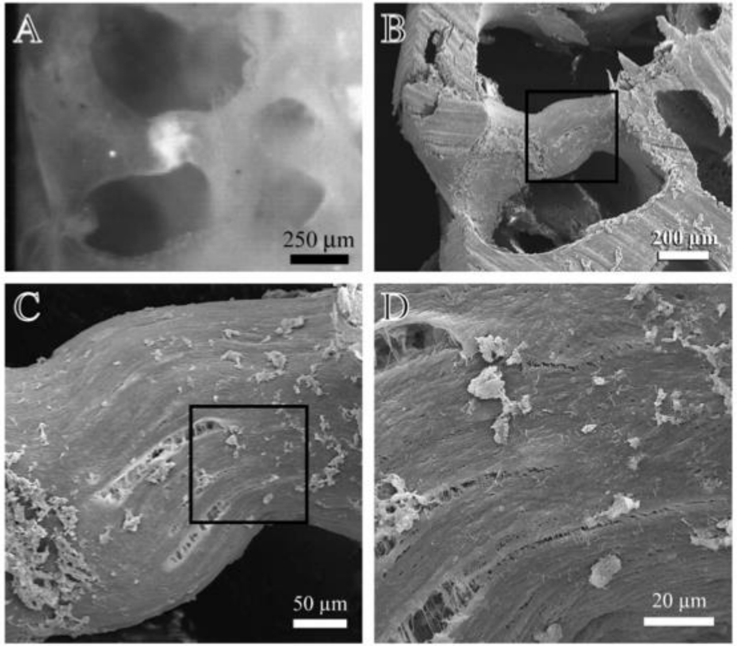
Stability: the ability of a structure to withstand a load without experiencing a sudden change in configuration

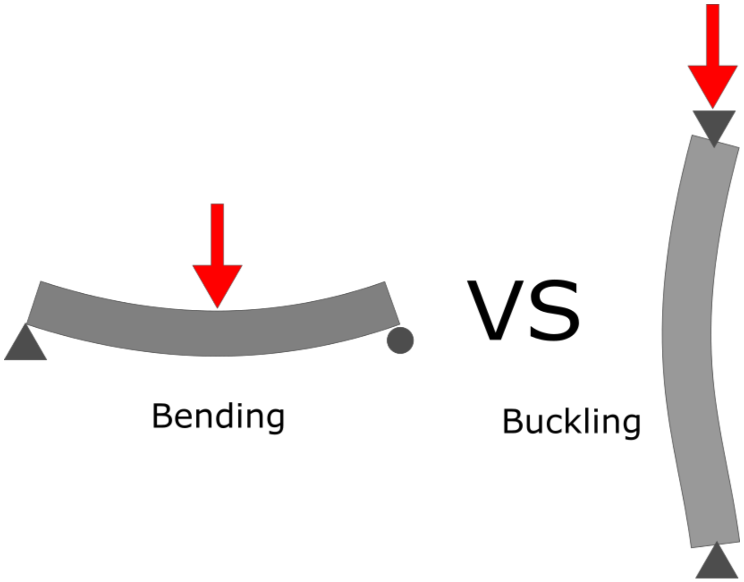
Buckling



Buckling is important from the macro to the microscale

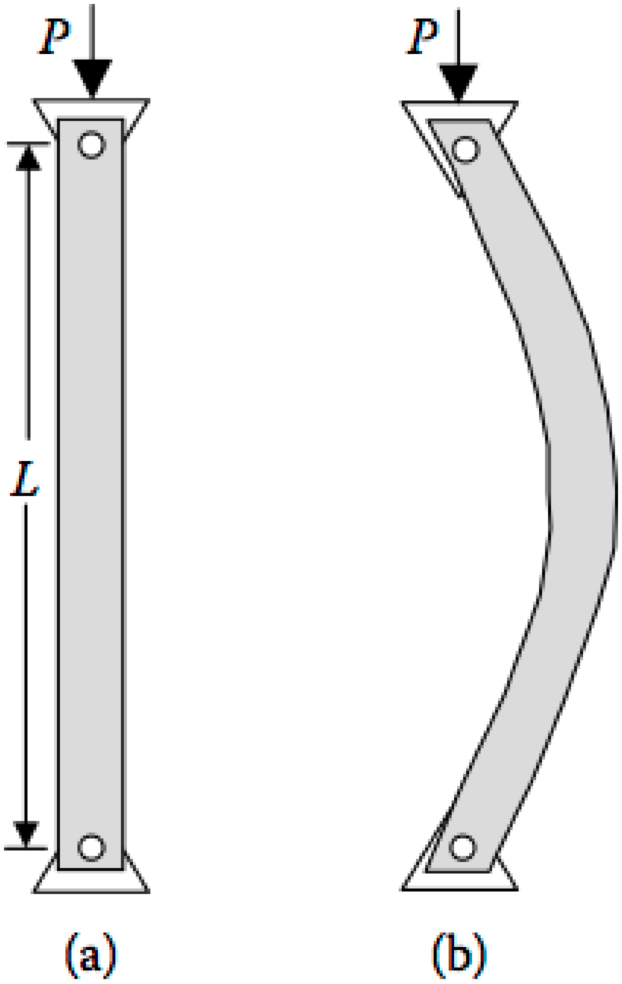






Buckling

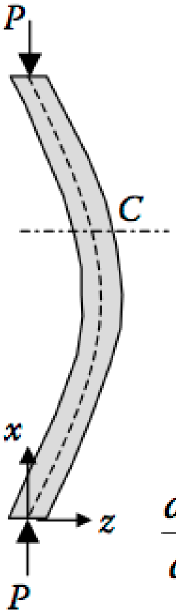
- Buckling is a type of instability that occurs when a beam fails under a compressive load much smaller than the load necessary to reach the yield stress
- In buckling, the failure occurs because the applied load results in a sudden deformation in a perpendicular direction.

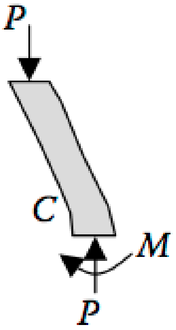


Euler Buckling

Two regimes of deformation when a beam is loaded in compression:

1. If the axial load on a beam is small, the change in length will be due to compressive strain. $P<P_{cr}$
2. If the axial load on a beam P is larger than the critical load $P_{cr,}$ then the beam becomes instable and a small perturbation will result in buckling of the beam.



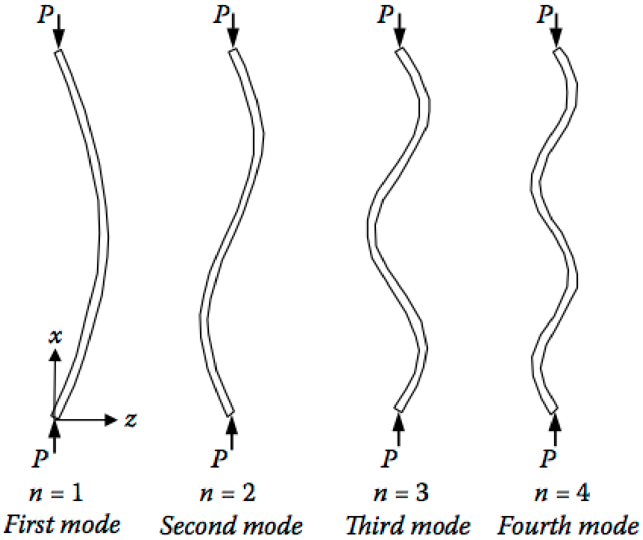


$$\frac{d^2w}{dx^2} = -\frac{M}{EI} = -\frac{P}{EI}w$$

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = 0$$

Euler Buckling

We can derive the Euler Buckling formula using the method of sections through the buckled beam.



Euler Buckling

The differential equation has multiple solutions:

This results in multiple bucking modes:

$$P = \frac{n^2 \pi^2 EI}{L^2}$$

Euler Formula

- For the critical buckling load we get then *Euler's Buckling Formula*:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \ .$$

- With the shape of the buckled beam:

$$w(x) = a \sin(\frac{\pi}{L} x)$$

- The second moment of area (I) should be taken around the axis around which the beam buckles. This is in general the axis with the smallest second moment of area.

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

Effective Length L_e as Function of Supports

End Conditions	Effective Length
Fixed-Free	$Le = 2L$
Pinned-Pinned	$Le = L$
Fixed-Pinned	$Le = 0.7L$
Fixed-Fixed	$Le = 0.5L$

Euler Buckling : Effective length

- The Euler Formula we have derived here only deals with a beam with pinned supports on each end.
- The type of support however greatly influences the critical load and the buckling behavior.
- Euler's formula can be extended towards other types of support by using the concept of the effective length.

Critical Buckling stress

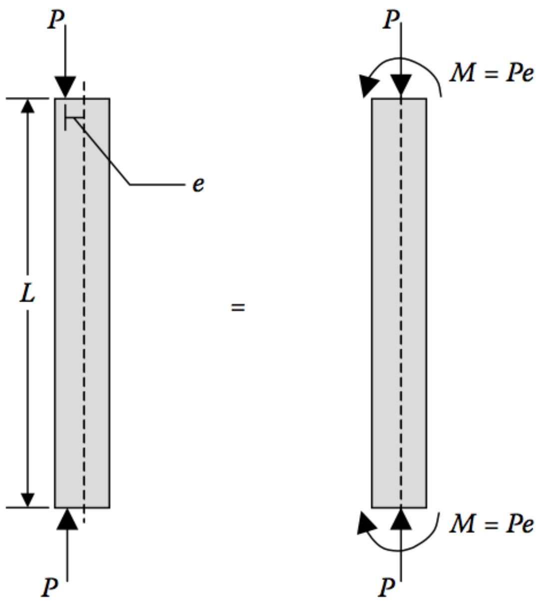
- Critical buckling stress:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{L_e^2 A}$$

- Using the definition of the radius of gyration $r = \sqrt{\frac{I}{A}}$:

$$\sigma_{cr} = \frac{\pi^2 EAr^2}{L_e^2 A} = \frac{\pi^2 Er^2}{L_e^2} = \frac{\pi^2 E}{(L_e / r)^2}$$

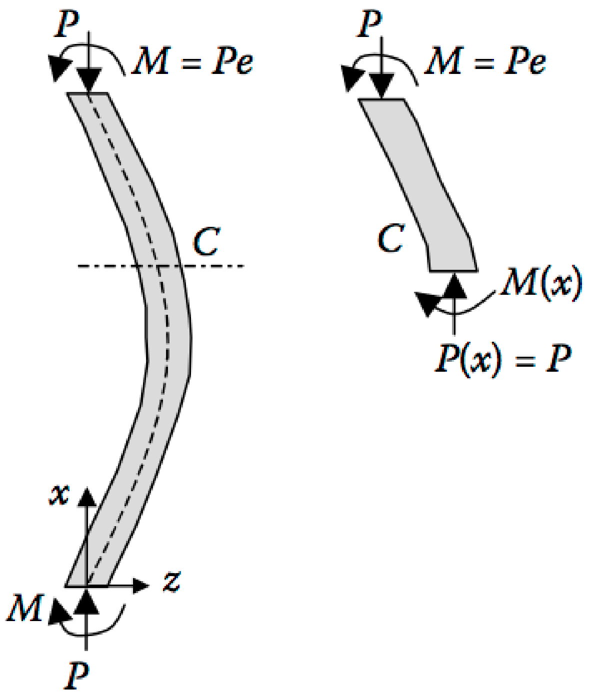
- L_e/r is the slenderness ratio



Buckling:
Effect of eccentricity

- So far, we've looked at beams that were loaded through the centroid of the column
- Often, the load is offset from this axis: eccentric loading
- We calculate the behavior of a beam pinned at both ends with eccentric load.
- We can model the eccentricity with an axial load and a moment at the supports

Buckling: Effect of eccentricity



- Centric load: P
- Moment: M=P*e
- This means that the beam bends even under small loads without the beam buckling
- We solve the now inhomogeneous differential equation:

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = -\frac{P}{EI} \cdot e$$

Buckling: Effect of eccentricity

- For the deflection we then get:

$$\boxed{w(x) = e\left\{\tan\left(\sqrt{\frac{P}{EI}}\cdot\frac{L}{2}\right)\cdot\sin\left(\sqrt{\frac{P}{EI}}\cdot x\right) + \cos\left(\sqrt{\frac{P}{EI}}\cdot x\right) - 1\right\}}$$

- And for the maximum deflection:

$$\boxed{w_{max} = w(L/2) = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) - 1\right]}$$

- The critical bucking load is then

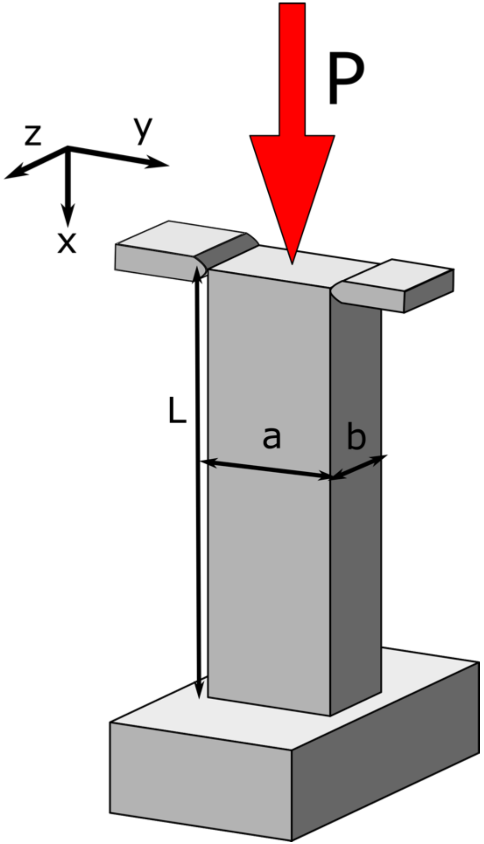
$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

Buckling: Effect of eccentricity

- Maximum stress in the beam is given by the compressive stress and the bending stress:

$$\boxed{\sigma_{max} = -\frac{P}{A}\left[1 - \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right]}$$

$$\boxed{\sigma_{max} = -\frac{P}{A}\left[1 - \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{P_{cr}}}\frac{\pi}{2}\right)\right]}$$



Example Buckling

An aluminum column of length *L* and rectangular cross section has a fixed end *B* and supports a centric axial load at *A*.

Two smooth and rounded fixed plates restrain end *A* from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

(a) Determine the ratio *a/b* of the two sides of the cross section corresponding to the most efficient design against buckling.

(b) Design the most efficient cross section for the column, knowing that *L* = 50 cm, *E* = 70 GPa, *P* = 22 kN and that a safety factor of 2.5 is required.